PRiME 2019 Final Report

Myles Ashitey, Brian Bishop, Kendall Bowens, Kayla Gibson,
Tayler Fernandes Nunez, and Cameron Thomas

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ABSTRACT

- Pt. 1 - A Belyĭ map $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a rational function with at most three critical values; we may assume these are $\{0,\, 1,\, \infty\}$. A Dessin d'Enfant is a planar bipartite graph on the sphere obtained by considering the preimage of a path between two of these critical values, usually taken to be the line segment from 0 to 1. Such graphs can be drawn on the sphere by composing with stereographic projection: $\beta^{-1}\big([0,1]\big) \subseteq \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$. This project sought to either create or expand on a database of such Belyĭ pairs, their corresponding Dessins d'Enfant, and their monodromy groups. We did so for up to degree N = 5 in the hopes of generating an algorithm to generate Dessins from monodromy triples.

- Pt. 2 - Arbitrarily choose loops γ around 0 and 1 in $\mathbb{P}^1\mathbb{C}$ that start and end at x_0. Compute the paths that start at P_k, where P_k is the k^{th} point that corresponds to the inverse image of x_0. We refer to these paths as $\widetilde{\gamma}$. Monodromy describes the movement of $\widetilde{\gamma}$ and γ in correspondence to a Belyĭ map (described in Project 1) such that the endpoints of our path correspond to a σ_0 and $\sigma_1 \in S_N$ where σ_∞ such that $\sigma_0 \circ \sigma_1 \circ \sigma_\infty = \mathbb{1}$ This project sought to simplify the concept of monodromy for a general audience in the form of an eight minute video. Our movie not only provides visualizations of monodromy on the Riemann Sphere but highlights monodromy's connection to Belyĭ maps and Dessin d'Enfant through real world examples.

This work is part of PRiME (Pomona Research in Mathematics Experience, NSF-1560394).

INTRODUCTION

During PRiME 2019, Dr. Edray Goins was the research mentor of a group of six undergraduates: Myles Ashitey (Pomona College), Brian Bishop (Pomona College), Kendall Bowens (Tuskegee University), Tayler Fernandes Nunez (Northeastern University), Kayla Gibson (University of Iowa), and Cameron Thomas

(Morehouse College) as well as one teaching assistant Alia Curtis (Scripps College). We worked on two projects during the Summer of 2019 from June 10th until July 31st.

In the first of the two projects, we aimed to add Dessins d'Enfant to an already existing database of Belyĭ Maps, monodromy permutation triples, and degree sequences. We expanded the database by adding the Dessins d'Enfant given Degree Sequences of genus 0 for $N \leq 5$. In addition to expanding the database, we constructed an algorithm that generates labeled Dessins d'Enfant on the plane given a monodromy permutation triple.

In the second project, we start with a bipartite graph $\Gamma = (V, E)$ embedded $\Gamma \hookrightarrow S^2(\mathbb{R})$ on the sphere such that its edges do not cross. This project sought to determine the monodromy groups of such graphs. We also discussed how to explain monodromy to a general audience using visualizations of group actions on the sphere. Furthermore, we produced a movie with such graphics.

All of this work was made possible due to the generous support of the NSF, Pomona College, and Dr. Edray Goins. We would like to express our deep gratitude to the NSF for funding the work we did this summer. We would also like to thank Pomona College for hosting us during this REU. Finally, thank you to Dr. Edray Goins for the fruitful mentorship.

BACKGROUND

Gennadiĭ Belyĭ proved in 1979 that a compact connected Riemann surface S of genus g is completely determined by the existence of a rational map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ which has three critical values. We say that a Belyĭ map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ is a rational map with critical values $\{0, 1, \infty\}$.

So what are "critical values"? Let's give an intuitive definition for $X = \mathbb{P}^1(\mathbb{C})$. Consider a function $\phi : X \to \mathbb{P}^1(\mathbb{C})$ A *critical point* $P \in X$ satisfies $\phi'(P) = 0$. A *critical value* $w \in \mathbb{P}^1(\mathbb{C})$ is $w = \phi(P)$ the value of a critical point P.

Following an idea from Alexander Grothendieck from 1984, we define a Dessin d'Enfant (French for "child's drawing") as a bipartite graph with "black" vertices $\beta^{-1}(0)$, "white" vertices $\beta^{-1}(1)$, midpoints of faces $\beta^{-1}(\infty)$, and edges $\beta^{-1}\big([0, 1]\big)$. For our purposes, these Dessins d'Enfant are projected onto the sphere using stereographic projection. The Dessin d'Enfants are projected onto the sphere using the following map:

$$\phi \cdot \mathbb{P}'(\mathbb{C}) \to S^2(\mathbb{R})$$

$$\phi([\tau_1 : \tau_0]) = \Big(\frac{2Re(\tau_1\overline{\tau_0})}{|\tau_1|^2 + |\tau_0|^2}r, \frac{2Im(\tau_1\overline{\tau_0})}{|\tau_1|^2 + |\tau_0|^2}r, \frac{|\tau_1|^2 - |\tau_0|^2}{|\tau_1|^2 + |\tau_0|^2}r\Big)$$

with radius r, a real number.

Now choose $P \in B \cup W \cup F$. We'll denote the *ramification index* e_p as the positive integer which is the number of edges at vertex P. The collection of the ramification indices can be collected into a multiset of multisets called the *degree sequence.* In 1891, Adolf Hurwitz showed the following four non-trivial properties:

i. The composition $\sigma_0 \circ \sigma_1 \circ \sigma_\infty = 1$ is the trivial permutation, and the subgroup $\mathrm{Mon}(\beta) = \langle \sigma_0, \sigma_1, \sigma_\infty \rangle$ of the symmetric group S_N generated by them is a transitive subgroup. This is called the *monodromy group* of β.

ii. Each of these permutations is a product of disjoint cycles:

$$\begin{aligned} \sigma_0 &= \prod_{P \in B} (b_{P,1}\ b_{P,2}\ \cdots\ b_{P,e_P}) & & B = \beta^{-1}(0) \\ \sigma_1 &= \prod_{P \in W} (w_{P,1}\ w_{P,2}\ \cdots\ w_{P,e_P}) & \text{corresponding to} \quad & W = \beta^{-1}(1) \\ \sigma_\infty &= \prod_{P \in F} (f_{P,1}\ f_{P,2}\ \cdots\ f_{P,e_P}) & & F = \beta^{-1}(\infty) \end{aligned}$$

where $e_P = \#\{Q \in S \,|\, \beta(Q) = \beta(P)\}$ is the ramification index of $P \in S$.

iii. The multiset $\mathcal{D} = \big\{\{e_P \,|\, P \in B\},\ \{e_P \,\big|\, P \in W\},\ \{e_P \,\big|\, P \in F\}\big\}$ is a collection of positive integers such that $N = \sum_{P \in B} e_P = \sum_{P \in W} e_P = \sum_{P \in F} e_P = |B| + |W| + |F| + (2\,g - 2)$.

iv. Conversely, any multiset $\mathcal{D}$ which a collection of three multisets is the degree sequence for some Belyĭ map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ if and only if there exist permutations $\sigma_0, \sigma_1, \sigma_\infty \in S_N$ such that the first three properties above hold.

In all, there is a $1-1$ correspondence between Belyĭ Maps $\beta :\to \mathbb{P}'(\mathbb{C})$, Dessins d'Enfant $\Delta_\beta \subseteq S$, and permutation triples $(\sigma_0, \sigma_1, \sigma_\infty)$. In fact, there are finitely many Belyĭ maps and Dessin d'Enfants for a given surface S of genus g and a given degree sequence N. We are motivated by this result to create a database in order to understand the relationship between these objects.

Outside from our own motivations, there seems to be growing interest in these topics throughout the STEM fields. For example, Belyĭ maps $\beta : S \to \mathbb{P}^1(\mathbb{C})$ are of interest to physicists due to the applications in String Theory, while monodromy groups $\mathrm{Mon}(\beta) = \langle \sigma_0, \sigma_1, \sigma_\infty \rangle$ are of interest to algebraic geometers and number theorists alike due to their relation with the Inverse Galois Problem.

The PRiME 2015 project sought (i) to compile and to create several examples of Belyĭ pairs and (ii) to create plots of their corresponding Dessins d'Enfant. The PRiME 2016 project sought (i) to start a database of Belyĭ pairs and (ii) to compute their monodromy groups. The PRiME 2017 project sought (i) to complete the database of Belyĭ pairs and their monodromy groups, (ii) to compute monodromy groups of compositions of Belyĭ maps, and (iii) to focus on monodromy groups of those Dessins d'Enfant which are toroidal graphs. Finally, the PRiME 2019 project sought (i)to expand the database of Belyĭ pairs and their monodromy groups with the addition of Dessins d'Enfant (ii) automate the process of computing Dessin d'Enfants given monodromy triples, and (iii) to focus on monodromy groups of those Dessins d'Enfant projected on the sphere.

PROJECT I

Let's begin by describing one of our first objectives: to expand the database of Belyĭ pairs and their monodromy groups with the addition of Dessins d'Enfant. Belyĭ Maps are peculiar functions that have undiscovered potential. By creating a database, we hope to gain a more complete understanding of them so we can harness said potential.

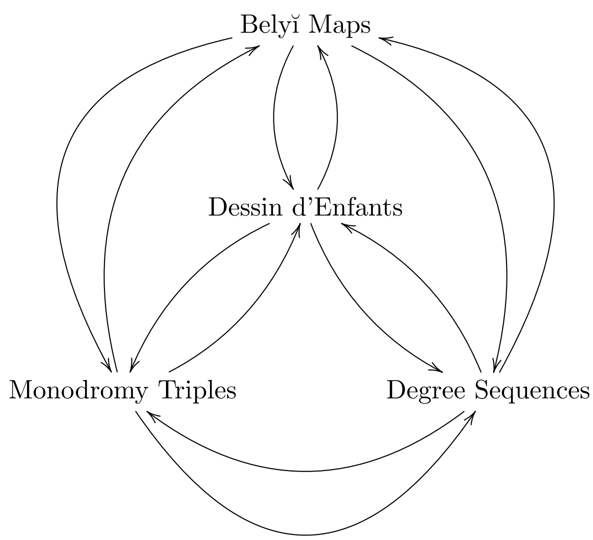


Figure 1: Objects in our Database

By looking at figure 1 above, we see the four parts of our database: Belyĭ maps, degree sequences, Dessins D'Enfant, and monodromy triples for a respective degree and genus. The pathways indicate that there is a certain correspondence between the elements of our database. What is important to keep in mind is there is a 1 to 1 correspondence between Belyĭ maps, monodromy triples, and Dessins d'Enfant, but not degree sequences which correspond to multiple of each object.

We want the database to include our four elements, recorded digitally, and easily navigable. To preface this, we did not find a final solution past wanting to make what was already available on the L-functions and Modular Forms Database (LMFDB) more complete and aesthetically pleasing. See LMFDB.org/Belyi/

Locating missing Belyĭ maps and Dessin d'Enfants required us to rely on entries complied in LMFDB. That database already contains the degrees, genera, degree sequences, and permutation triples for Belyĭ maps past genus 0. It is incredibly thorough, with the directory being fairly straightforward. We were able to locate the Belyĭ maps we wanted using criterion like its genus or degree. The database falls short in its attempt at intuitiveness. The more we navigated throughout the website, the more this issue became apparent. In fact, we found the classification of the Belyĭ maps confusing. LMFDB uses a label that combines the degree, genus, degree sequence, and size of the group to categorize its Belyĭ maps. When scrolling through the list of Belyĭ maps, it was unclear as to what each aspect of the classifier meant. LMFDB's greatest shortcoming was the omission of Dessin d'Enfant from its database of Belyĭ maps. Dessins d'Enfant have a clear connection to Belyĭ maps, permutation triples, and degree sequences. The visualization given through a Dessin d'Enfant provides clarity to aspects of monodromy that may appear overwhelming at first glance. We looked to remedy this issue with the creation of our own database. Our first step towards creating a database was to draw our own Dessins from a given set of monodromy triples and degree sequences. This was basically just a check of the database that already existed since all of this information was available for N<4.

While going through this drawing and validating process, something we found particularly interesting was that there are two ways to draw a Dessin for the same monodromy triple if you perform a simple transformation of the labeling and the orientation of edges of a Dessin. Here is an example:

$$D = \big\{\{2,3\},\{1,1,3\},\{2,3\}\big\}$$

$$\sigma_0 = (35)(142)\ \sigma_1 = (2)(3)(145)\ \sigma_\infty = (12)(345)$$

The monodromy triple and degree sequence above correspond to the two Dessins d'Enfant in Figure 2.

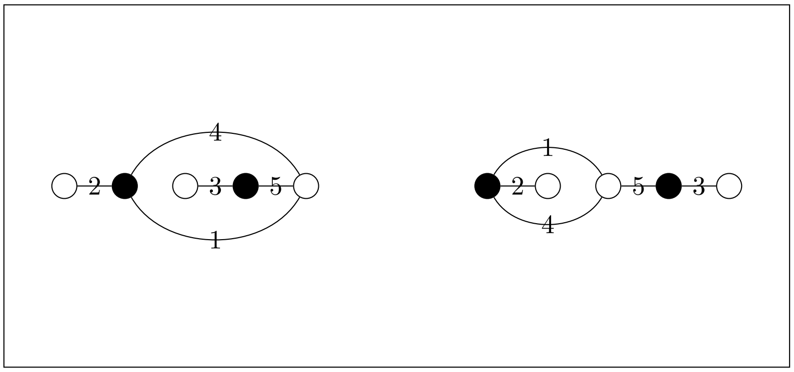


Figure 2: Two Dessins d'Enfant of the same triple & degree sequence

Notice how both of these Dessins are valid interpretations of the given monodromy Triples. A possible reason for this is if we were to project these onto the sphere they would result in the same Dessin, however this fact needs more exploration.

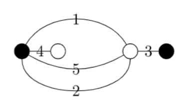
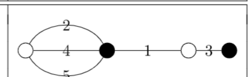
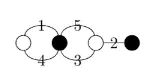
The next step was to electronically record our Dessins. In the case where we found multiple Dessins given a monodromy triple and degree sequence, we decided to go with only one of the options, assuming that multiple drawings still correspond to the same Dessin. Also, since one degree Sequence might correspond to multiple monodromy triples (and the subsequent 1 to 1 correspondence of Belyĭ maps, monodromy triples, and Dessins), we chose to only record one iterated degree sequence to represent all of its other possibilities. We used LaTeX in conjunction with TikZ package to draw these Dessins for this first draft of our database. Since LaTeX is a typesetting system, it was very difficult to add to our database and make frequent changes. This is where our intuition to create code that streamlines this process became more apparent.

The next step was to utilize the tools given to us to fill out the missing information in our database. We had a program from Dr. Goins called Dessin Explorer which would plot a Dessin d'Enfant on the complex plane after inputing a Belyĭ map. That in conjunction with the Belyĭ maps found on LMFDB allowed us to verify three Dessins d'Enfant for the Belyĭ maps of genus 0 and N=4, along with the fourteen Dessins of genus 0 and N=5, both of which were missing from Dr. Goins' database. The first task proved relatively simple because we were given the Belyĭ maps that would be plugged into the Dessin Explorer. It was more difficult to verify the Dessin d'Enfants of N=5 where only two of these Dessins d'Enfant had Belyĭ maps paired with them and twelve did not.

For the two Dessins d'Enfant that had Belyĭ maps, we were able to verify by way of the Dessin Explorer. Verifying the twelve remaining Dessins was tedious. We began by creating a separate database for N=5. It contained the permutations, degree sequences, and Belyĭ maps found on LMFDB. We entered everything as it appeared on the website. We created Dessins corresponding to the information found on LMFDB and attempted to verify it using the Dessin Explorer. With some of the Belyĭ maps, we ran into the issue of not being able to generate the Dessin. The program was designed to generate our Dessins, all of which are on the complex plane and require the point at infinity. We needed to apply a Möbius transformation to the Dessins on the LMFDB database, switching the point at 0 and the point at ∞. Applying this transformation also meant we needed to switch each σ_0 and σ_∞, the first and last value of each degree sequence, and the numerator and denominator of each Belyĭ map that failed to run in the Dessin Explorer. After applying the Möbius transformation, the Dessin d'Enfant that the program yielded matched with those that were sketched out earlier. We were able to successfully verify the Dessins of degree 5.

We labeled our second database Database V2. It contained all the missing Dessins, missing Belyĭ maps, permutation triples, and degree sequences for N=5, genus 0 as well as the missing Dessins, Belyĭ maps, permutation triples, and degree sequences for N=4, genus 0.

At higher degrees of N, the Belyĭ maps become fairly lengthy. As a result, we ran into the issue of deciding how we wanted to format the database. One possibility was to place each Belyĭ map a row below the genus, degree, degree sequence, permutation triple, and Dessin d'Enfant it corresponded to. We opted to create an appendix for Belyĭ maps of $N = 5$, genus 0.

Degree N	Genus g	Degree Sequence $\mathcal{D}$	Monodromy Triple	Belyĭ Map	Dessin d'Enfant
$N = 5$	$g = 0$	$\mathcal{D} = \{\{4,1\},\ \{4,1\},\ \{2,2,1\}\}$	$\sigma_0 = (1\ 2\ 5\ 4)(3)$ $\sigma_1 = (1\ 5\ 2\ 3)\,(4)$ $\sigma_\infty = (1\ 4)\,(2\ 3)\,(5)$	$\beta(z) = \#12$	
$N = 5$	$g = 0$	$\mathcal{D} = \{\{4,1\},\ \{3,2\},\ \{3,1,1\}\}$	$\sigma_0 = (1\ 2\ 4\ 5)\,(3)$ $\sigma_1 = (1\ 3)\,(2\ 5\ 4)$ $\sigma_\infty = (1\ 2\ 3)\,(4)\,(5)$	$\beta(z) = \#13$	
$N = 5$	$g = 0$	$\mathcal{D} = \{\{4,1\},\ \{3,2\},\ \{3,1,1\}\}$	$\sigma_0 = (1\ 4\ 3\ 5)\,(2)$ $\sigma_1 = (1\ 4)\,(2\ 5\ 3)$ $\sigma_\infty = (1\ 2\ 3)\,(4)\,(5)$	$\beta(z) = \#14$	

12. $\beta(z) = \frac{1}{4107}\frac{(-136i+4623)x^5+(-15096i+513153)x^4}{x^5+(40i+55)x^4+(13236i-12048)x^3+(1006992i-709956)x^2+(-67346586i-36186777)x-7475471046i-4016732247}$

13. $\beta(z) = \frac{(-5568\sqrt{6}+14338)x^5+(153\sqrt{6}-1448)x^4}{5634x^5+(-2583\sqrt{6}-9432)x^4+(-2304\sqrt{6}-5076)x^3+(6686\sqrt{6}+16344)x^2+(11912\sqrt{6}+29178)x-18283\sqrt{6}-44784}$

14. $\beta(z) = \frac{(-5568\sqrt{6}+14338)x^5+(153\sqrt{6}-1448)x^4}{5634x^5+(-2583\sqrt{6}-9432)x^4+(-2304\sqrt{6}-5076)x^3+(6686\sqrt{6}+16344)x^2+(11912\sqrt{6}+29178)x-18283\sqrt{6}-44784}$

Now let's cover another motivation for Project I: to generate the Dessins d'Enfant from monodromy triples. Generating these bipartite graphs by hand can become extremely taxing, tedious and may take a while for higher degree N. For this reason, we were motivated to automate the process of producing Dessins d'Enfant from monodromy permutation triples. The creation of this process or algorithm, started with looking at the type of information the monodromy triples provided. Let's look at σ_0, where σ_0 is the product of disjoint cycles in which each cycle is associated to a black vertex. The numbering inside each cycle is the counterclockwise ordering of edges connected to each black vertex. Similarly, σ_1 is the product of disjoint cycles where each cycle is associated to a white vertex. The numbering inside each cycle is the counterclockwise ordering of each edge coming out of a white vertex. We were able to use this information in order to construct the steps of our algorithm. To graph the Dessins d'Enfant, we needed to find which vertices are connected. To find this correspondence, we started by labeling each cycle 1 through N, where N equals the total amount of cycles in σ_0 and σ_1. Thus, vertices labeled "1" to "$| \sigma_0 |$" are the black vertices, and the vertices labeled "$| \sigma_0 | +1$" to "$| \sigma_0 | + | \sigma_1 |$" are the white vertices. Then we plotted each vertex with the correct counterclockwise labeling of edges and connected black and white vertices if they had an edge in common. While following the steps of this algorithm, we were careful to keep the correct counterclockwise ordering of edges and made sure that edges did not cross. For example, let $\sigma_0 = (12)$ and $\sigma_1 = (1)(2)$. Following our algorithm, the cycle (1 2) corresponds to a black vertex labeled "1." The following cycle (1), from σ_1, corresponds to a white vertex labeled "2." Finally, cycle (2) corresponds to a white vertex labeled "3." Since the cycle (1 2) and the cycle (1) share the number one, the vertices "1" and "2" are connected. After connecting the labeled black and white vertices based on the edges they share, those edges are labeled based on said connection. In other words, the two vertices "1" and "2" share the number one in their cycle, thus they are connected by an edge labeled "1". Similarly, vertices "1" and "3" are connected because they share the number two in their cycles; furthermore, the two vertices will be connected by an edge labeled "2". Thus, the Dessin d'Enfant for this monodromy triple looks like the graph in figure 3.

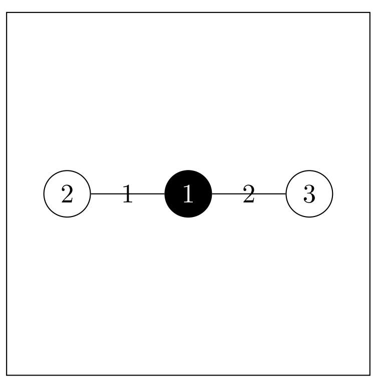


Figure 3: Dessin d'Enfant for $\sigma_0 = (12)$, $\sigma_1 = (1)(2)$, $\sigma_\infty = (12)$

After constructing the algorithm, we wanted to implement it using Cocalc as our platform. While working on the code in Sage, we compared the outputs to a list of Dessins we had drawn. The code worked for most of the simpler graphs, but we noticed that more complex graphs were not maintaining their orientation and the edges were not maintaining their cyclic order. To fix this problem, we tried to use the set embedding command. However, the command did not work in Sage. Therefore, we tried to rewrite the code in Mathematica, yet the code still produced the same error. Another problem we encountered was that Sage and Mathematica had trouble handling graphs that contained multiple edges. After facing these complications, we identified areas of improvement for future implementation. The first step towards improvement is to add a command that successfully maintains one orientation for each Dessin along with maintaining the cyclic order of edges. Another step would be to write a function inside the code that checks if the graph has the appropriate amount of faces.

PROJECT II

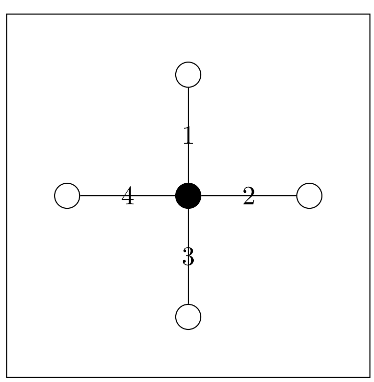
Given the diagram in figure 1, we know that no matter where we start, there is an algorithm corresponding to a pathway that will get us to another part. Project II focused on monodromy groups; how to compute them from Dessin d'Enfants and Belyĭ maps, and how they are used to draw Dessins d'Enfant.

Define a monodromy group as a permutation triple $\sigma_0, \sigma_1, \sigma_\infty \in S_N$ This group will satisfy the following properties

- $\sigma_0 \circ \sigma_1 \circ \sigma_\infty = \mathbb{1}$ is the trivial permutation
- $\mathrm{Mon}(\beta) =< \sigma_0, \sigma_1, \sigma_\infty >$ is a transitive subgroup

It's important to note that $\sigma_0, \sigma_1, \sigma_\infty$ correspond to the movement around the black vertices, white vertices, and midpoints of faces, respectively. Each vertex coincides to its own disjoint cycle. An example goes as follows.

$\sigma_0 = (1432)$
$\sigma_1 = (1)(2)(3)(4)$
$\sigma_\infty = (1234)$



Label the edges 1 through 4 and loop counterclockwise around black and white vertices. $|B|$ represents the number of disjoint cycles in σ_0, $|W|$ represents the number of disjoint cycles in σ_1, and $|F|$ represents the number of disjoint cycles in σ_∞. The length of each disjoint cycle or the number of edges coming out of each vertex is e_P. The integers inside each disjoint cycle is the labeling of the edges in counterclockwise order.

We can compute the permutation triples $\sigma_0, \sigma_1, \sigma_\infty$ from a given Belyĭ map $\beta : S \to \mathbb{P}^1(\mathbb{C})$ with the following five steps:

1. Choose $x_0 \neq 0, 1, \infty$ and compute the inverse image of x_0. In other words, compute: $\beta^-1(x_0) = \{P_1, P_2, ...P_N\}$.

2. Choose loops γ around 0 and 1 in $\mathbb{P}^1\mathbb{C}$ that start and end at x_0. An example of a loop is $\gamma_\epsilon(t) = \epsilon + (x_0 - \epsilon)e^{2\pi i t}$.

3. Compute the paths that start at P_k. We can refer to these functions, or paths as $\widetilde{\gamma}$. By this definition we have that $\beta \circ \widetilde{\gamma}_\epsilon^{(k)} = \gamma_\epsilon$ and $\widetilde{\gamma}_\epsilon^{(k)}(0) = P_k$.

4. Compute σ_0 and $\sigma_1 \in S_N$ using the endpoints of these paths. In other words, compute $\widetilde{\gamma}_\epsilon^{(k)}(1) = P_{\sigma\,\epsilon}(k)$.

5. Finally, we compute σ_∞ such that $\sigma_0 \circ \sigma_1 \circ \sigma_\infty = \mathbb{1}$

The goal of Project II was to explain the steps outlined above to a general audience. In order to explain this concept to anyone despite their level of mathematical knowledge, we decided to produce a movie. We met as a group to discuss the goals we wanted to achieve. These goals included making sure the content was concise and comprehensible, coupled with graphics, analogies and humor to keep the movie interesting. Keeping these goals in mind, we were influenced by characteristics from different math YouTube channels to create our movie. Once the brainstorming was done, we divided the movie into five sections: Wow, Intro, Stereographic Projection, Monodromy, and Outro. Finally, we planned out the content for each section and created a rough draft of a storyboard.

We divided ourselves to work on sections and two people were tasked with editing. Since we decided to use styles from different YouTube channels, we went ahead and used different techniques to create various clips. For our introduction, we used raw footage and animation. As for the stereographic projection segment, we used pens and paper to create a speed drawing segment. Once the footage was shot, the team members that filmed their respective sections of the movie sat down with the editors. We used a myriad of programs to edit our movie such as Adobe Premiere as our primary editing software, Adobe Illustrator to create our vector image for the movie, both Adobe Photoshop and Blender to create our animations for the movie, Mathematica to generate and render the monodromy clips, and Adobe Lightroom to edit photos to be more aesthetically pleasing. Through these programs, we were able to create the movie we envisioned.

In our research this summer, we developed a movie that explains monodromy on the sphere. Moving forward, we would also like to create a movie of monodromy on the torus that is easily accessible to mathematicians and academics. While we have developed the start to an algorithm in CoCalc that accepts σ_0 and σ_1 and outputs the corresponding Dessin d'Enfant, we would like to complete the our work with code that works consistently. In the future, we would like to utilize a functional "set-embedding" command to finalize our algorithm. Our goal is to have our research be available to the public in the form of a website that would allow users to view Dessins d'Enfant on the unit sphere and on other Riemann surfaces of higher genera. In addition, this website would allow users to create movies that would display loops around critical values with corresponding paths. Finally, while LMFDB contains information on certain Belyĭ maps, our website would also expand our database through the utilization of our algorithm that create Dessins d'Enfant from existing monodromy groups. All of this contributes to our ultimate goal of being able to access data for Belyĭ maps, Dessins d'Enfant, and degree sequences, given a monodromy group.

REFERENCES

N. M. Adrianov, N. Ya Amburg, V. A. Dremov, Yu A. Levitskaya, E. M. Kreines, Yu. Yu. Kochetkov, V. F. Nasretdinova, and G. B. Shabat. Catalog of dessins d'enfants with $\leq 4edges.\texttt{arXiv.org},0710.2658v1(2007).64pages.$

G. V. Belyi. On extensions of the maximal cyclotomic field having a given classical Galois group. *J. reine angew. Math.*, 341 (1983), 147-156.

A. Grothendieck. Esquisse d'un programme. *Rapport Scientifique*, 1984.

A Hurwitz. Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten. *Mathematische Annalen*, 39(1):1–60, 1891.

Numerical calculation of three-point branched covers of the projective line. `arXiv.org`, 1311.2081v3 (2013). 58 pages.

A Database of Belyĭ Maps. `arXiv.org`, 1805.07751v2 (2018).

Leonardo Zapponi. What is . . . a Dessin d'Enfant? *Notices Amer. Math. Soc.*, (2003), 788–798.